MATH GU 4156 **Advanced Probability Theory** Columbia University (Fall 2023)

Lecture 7: Uniform Integrability

Lecturer: ***Ioannis Karatzas*** *Scribes: Heyuan Yao*

Let us place ourselves on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $X : \omega \to \mathbb{R}$ is integrable, then the DCT gives

$$\mathbb{E}(|X|\mathbb{1}_{\{|X|>\lambda\}}) \to 0, \quad \lambda \to \infty.$$

Here is a generalization of this observation.

Definition 7.1 *A family $\{X_\alpha\}_{\alpha \in A}$ of random variables is called* ***uniformly integrable****, if*

$$\sup_{\alpha \in A} \mathbb{E}\left(\mathbb{1}_{|X_\alpha|>\lambda}\right) \to 0, \quad \lambda \to \infty.$$

This notion is crucial. It is weaker than boundedness in $\mathbb{L}^p$ for $p > 1$, but stronger than boundedness in $\mathbb{L}^1$.

Example: On $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathbb{B}, Leb)$ consider $X_n(\omega) = n \cdot \mathbb{1}_{(0,\frac{1}{n})(\omega)}, n \in \mathbb{N}_0$. Clearly $\mathbb{E}|X_n|= 1$, $\forall n \in \mathbb{N}_0$, so this sequence is bounded in $\mathbb{L}^1$. However, with $n \geq \lambda > 0$ we have

$$\mathbb{E}\left(|X_n|\mathbb{1}_{\{|X_n|>\lambda\}}\right) = nLeb(0, \frac{1}{n}) = 1,$$

so this sequence is not uniformly integrable.

Example: De La Vallée Poussin Criterion: Suppose that $\sup_{\alpha \in A} \mathbb{E}[h(|X_\alpha|)] < \infty$ holds for some function $h : (0\infty) \to (0\infty)$ with $h(0+) = 0$ and such that $x \mapsto \frac{h(x)}{x}$ is increasing, with $\lim_{x \to \infty} \frac{h(x)}{x} = \infty$. then the family $\{X_\alpha\}_{\alpha \in A}$ is uniformly integrable.

The great mathematician Charles-Jean Étienne Gustave Nicolas, Baron De La Vallée Poussin, came up with this criterion; and showed that, conversely, if $\{X_\alpha\}_{\alpha \in A}$ is uniformly integrable, there exists a function $h : (0\infty) \to (0\infty)$ with these properties.

We have also the following characterization for this notion.

Proposition 7.2 *A family of random variable $\{X_\alpha\}_{\alpha \in A}$ is uniformly integrable if, and only if, both conditions below hold:*

(i) Boundedness in $\mathbb{L}^1$: $\sup_{\alpha \in A} \mathbb{E}(|X_\alpha|) < \infty$.

(ii) Uniform Absolute Continuity:

$(\forall \epsilon > 0)(\exists \delta_\epsilon > 0)$ *s.t.* $\sup_{\alpha \in A} \mathbb{E}(|X_\alpha|\mathbb{1}_B) < \epsilon$ *holds for every* $B \in \mathcal{F}$ *with* $\mathbb{P}(B) < \delta_\epsilon$.

We know very well by now, that a sequence of integrable random variables can easily converge a.e., but fail to converge in $\mathbb{L}^1$ The following result identifies uniform integrability as the "unique glue" that makes this happen.

Theorem 7.3 (Generalized DCT) *Suppose the sequence of integrable r.v.'s $X_1, X_2 \ldots$ converges in probability to some random variable X.*

Then the following are equivalent:

(i) *the $X_1, X_2, \ldots$ are uniformly integrable;*

(ii) $\mathbb{E}(|X_n - X|) \stackrel{n\to\infty}{\to} 0$*, i.e.,* $X_n \stackrel{\mathbb{L}^1}{\to} X$*;*

(iii) $\mathbb{E}|X_n| \stackrel{n\to\infty}{\to} \mathbb{E}|X|$.

The equivalence (i) and (ii) above, has become known as SCHEFFÉ's lemma.

Finally, we have a famous compactness result.

Theorem 7.4 (DUNFORD-PETTIS) *Consider a family $\{X_\alpha\}_{\alpha\in A}$ of integrable random variables. Then the following are equivalent:*

(i) *the family is uniformly integrable.*

(ii) *every sequence $(X_n)_{n\in\mathbb{N}_0}$ from the family, contains a subsequence which converges weakly in $\mathbb{L}^1$ to some $X \in \mathbb{L}^1$.*